Chapter 2

Essentials of Linear Viscoelasticity

In this chapter the fundamentals of the linear theory of viscoelasticity are presented in the one-dimensional case. The classical approaches based on integral and differential constitutive equations are reviewed. The application of the Laplace transform leads to the so-called *material functions* (or step responses) and their (continuous and discrete) time spectra related to the creep and relaxation tests. The application of the Fourier transform leads to the so-called *dynamic functions* (or harmonic responses) related to the storage and dissipation of energy.

2.1 Introduction

We denote the stress by $\sigma = \sigma(x,t)$ and the strain by $\epsilon = \epsilon(x,t)$ where x and t are the space and time variables, respectively. For the sake of convenience, both stress and strain are intended to be normalized, i.e. scaled with respect to a suitable reference state $\{\sigma_* \,, \epsilon_*\}$.

At sufficiently small (theoretically infinitesimal) strains, the behaviour of a viscoelastic body is well described by the linear theory of viscoelasticity. According to this theory, the body may be considered as a linear system with the stress (or strain) as the excitation function (input) and the strain (or stress) as the response function (output).

To derive the most general stress–strain relations, also referred as the *constitutive equations*, two fundamental hypotheses are required: (i) invariance for time translation and (ii) causality; the former means that a time shift in the input results in an equal shift in the output, the latter that the output for any instant t_1 depends on the values of the input only for $t \le t_1$. Furthermore, in this respect, the response functions to an excitation expressed by the unit step function $\Theta(t)$, known as Heaviside function defined as

$$\Theta(t) = \begin{cases} 0 & \text{if } t < 0\,, \\ 1 & \text{if } t > 0\,, \end{cases}$$

are known to play a fundamental role both from a mathematical and physical point of view.

The creep test and the relaxation test. We denote by $J(t)$ the strain response to the unit step of stress, according to the *creep test*

$$\sigma(t) = \Theta(t) \Longrightarrow \epsilon(t) = J(t)\,, \tag{2.1a}$$

and by $G(t)$ the stress response to a unit step of strain, according to the *relaxation test*

$$\epsilon(t) = \Theta(t) \Longrightarrow \sigma(t) = G(t)\,. \tag{2.1b}$$

The functions $J(t)$ and $G(t)$ are usually referred as the *creep compliance* and *relaxation modulus* respectively, or, simply, the *material functions* of the viscoelastic body. In view of the causality requirement, both functions are *causal*, i.e. vanishing for $t < 0$. Implicitly, we assume that all our causal functions, including $J(t)$ and $G(t)$, are intended from now on to be multiplied by the Heaviside function $\Theta(t)$.

The limiting values of the material functions for $t \to 0^+$ and $t \to +\infty$ are related to the instantaneous (or glass) and equilibrium behaviours of the viscoelastic body, respectively. As a consequence, it is usual to set

$$\begin{cases} J_g := J(0^+) & \textit{glass compliance}\,, \\ J_e := J(+\infty) & \textit{equilibrium compliance}\,; \end{cases} \tag{2.2a}$$

and

$$\begin{cases} G_g := G(0^+) & \textit{glass modulus}\,, \\ G_e := G(+\infty) & \textit{equilibrium modulus}\,. \end{cases} \tag{2.2b}$$

From experimental evidence, both the material functions are non-negative. Furthermore, for $0 < t < +\infty$, $J(t)$ turns out to be a non-decreasing function, whereas $G(t)$ a non-increasing function. Assuming that $J(t)$ is a *differentiable, increasing function of time*, we write

$$t \in \mathbb{R}^+ , \; \frac{dJ}{dt} > 0 \implies 0 \leq J(0^+) < J(t) < J(+\infty) \leq +\infty \, . \quad (2.3a)$$

Similarly, assuming that $G(t)$ is a *differentiable, decreasing function of time*, we write

$$t \in \mathbb{R}^+ , \; \frac{dG}{dt} < 0 \implies +\infty \geq G(0^+) > G(t) > G(+\infty) \geq 0 \, . \quad (2.3b)$$

The above characteristics of monotonicity of $J(t)$ and $G(t)$ are related respectively to the physical phenomena of *strain creep* and *stress relaxation*, which indeed are experimentally observed. Later on, we shall outline more restrictive mathematical conditions that the material functions must usually satisfy to agree with the most common experimental observations.

The creep representation and the relaxation representation. Hereafter, by using the Boltzmann superposition principle, we are going to show that the general stress – strain relation is expressed in terms of one material function [$J(t)$ or $G(t)$] through a linear hereditary integral of Stieltjes type. Choosing the *creep representation*, we obtain

$$\epsilon(t) = \int_{-\infty}^{t} J(t-\tau)\, d\sigma(\tau) \, . \quad (2.4a)$$

Similarly, in the *relaxation representation*, we have

$$\sigma(t) = \int_{-\infty}^{t} G(t-\tau)\, d\epsilon(\tau) \, . \quad (2.4b)$$

In fact, since the responses are to be invariant for time translation, we note that in $J(t)$ and $G(t)$, t is the time *lag* since application of stress or strain. In other words, an input $\sigma(t) = \sigma_1\, \Theta(t-\tau_1)$ [$\epsilon(t) = \epsilon_1\, \Theta(t-\tau_1)$] would be accompanied by an output $\epsilon(t) = \sigma_1\, J(t-\tau_1)$ [$\sigma(t) = \epsilon_1\, G(t-\tau_1)$]. As a consequence, a series of N stress steps $\Delta\sigma_n = \sigma_{n+1} - \sigma_n$ $(n = 1, 2, \ldots, N)$ added consecutively at times

$$\tau_N > \tau_{N-1} > \cdots > \tau_1 > -\infty \, ,$$

will induce the total strain according to

$$\sigma(t) = \sum_{n=1}^{N} \Delta\sigma_n \, \Theta(t-\tau_n) \implies \epsilon(t) = \sum_{n=1}^{N} \Delta\sigma_n \, J(t-\tau_n)\,.$$

We can approximate arbitrarily well any physically realizable stress history by a step history involving an arbitrarily large number of arbitrarily small steps. By passing to the limit in the sums above, we obtain the strain and stress responses to arbitrary stress and strain histories according to Eqs. (2.4a) and (2.4b), respectively. In fact

$$\sigma(t) = \int_{-\infty}^{t} \Theta(t-\tau)\, d\sigma(\tau) = \int_{-\infty}^{t} d\sigma(\tau) \implies \epsilon(t) = \int_{-\infty}^{t} J(t-\tau)\, d\sigma(\tau)\,,$$

and

$$\epsilon(t) = \int_{-\infty}^{t} \Theta(t-\tau)\, d\epsilon(\tau) = \int_{-\infty}^{t} d\epsilon(\tau) \implies \sigma(t) = \int_{-\infty}^{t} G(t-\tau)\, d\epsilon(\tau)\,.$$

Wherever the stress [strain] history $\sigma(t)$ $[\,\epsilon(t)\,]$ is differentiable, by $d\sigma(\tau)$ $[\,d\epsilon(\tau)\,]$ we mean $\dot\sigma(\tau)\,d\tau$ $[\dot\epsilon(\tau)\,d\tau\,]$, where we have denoted by a superposed dot the derivative with respect to the variable τ. If $\sigma(t)$ $[\,\epsilon(t)\,]$ has a jump discontinuity at a certain time τ_0, the corresponding contribution is intended to be $\Delta\sigma_0\, J(t-\tau_0)$ $[\,\Delta\epsilon_0\, G(t-\tau_0)\,]$.

All the above relations are thus a consequence of the Boltzmann superposition principle, which states that in linear viscoelastic systems the total response to a stress [strain] history is equivalent (in some way) to the sum of the responses to a sequence of incremental stress [strain] histories.

2.2 History in $\mathbb{R}^+$: the Laplace transform approach

Usually, the viscoelastic body is quiescent for all times prior to some starting instant that we assume as $t = 0$; in this case, under the hypotheses of causal histories, differentiable for $t \in \mathbb{R}^+$, the creep and relaxation representations (2.4a) and (2.4b) reduce to

$$\epsilon(t) = \int_{0^-}^{t} J(t-\tau)\, d\sigma(\tau) = \sigma(0^+)\, J(t) + \int_{0}^{t} J(t-\tau)\, \dot\sigma(\tau)\, d\tau\,, \quad (2.5a)$$

$$\sigma(t) = \int_{0^-}^{t} G(t-\tau)\, d\epsilon(\tau) = \epsilon(0^+)\, G(t) + \int_{0}^{t} G(t-\tau)\, \dot\epsilon(\tau)\, d\tau\,. \quad (2.5b)$$

Unless and until we find it makes any sense to do otherwise, we implicitly restrict our attention to causal histories.

Another form of the constitutive equations can be obtained from Eqs. (2.5a) and (2.5b) by integrating by parts. We thus have

$$\epsilon(t) = J_g\, \sigma(t) + \int_0^t \dot{J}(t-\tau)\, \sigma(\tau)\, d\tau \,, \tag{2.6a}$$

and, if $G_g < \infty$,

$$\sigma(t) = G_g\, \epsilon(t) + \int_0^t \dot{G}(t-\tau)\, \epsilon(\tau)\, d\tau \,. \tag{2.6b}$$

The causal functions $\dot{J}(t)$ and $\dot{G}(t)$ are referred as the *rate of creep (compliance)* and the *rate of relaxation (modulus)*, respectively; they play the role of *memory functions* in the constitutive equations (2.6a) and (2.6b). If $J_g > 0$ or $G_g > 0$ it may be convenient to consider the non-dimensional form of the memory functions obtained by normalizing them to the glass values[1].

The integrals from 0 to t in the R.H.S of Eqs. (2.5a) and (2.5b) and (2.6a) and (2.6b) can be re-written using the convolution form and then dealt with the technique of the Laplace transforms, according to the notation introduced in Chapter 1,

$$f(t)\, *\, g(t) \div \widetilde{f}(s)\, \widetilde{g}(s)\,.$$

Then, we show that application of the Laplace transform to Eqs. (2.5a) and (2.5b) and (2.6a) and (2.6b) yields

$$\widetilde{\epsilon}(s) = s\, \widetilde{J}(s)\, \tilde{\sigma}(s)\,, \tag{2.7a}$$

$$\tilde{\sigma}(s) = s\, \widetilde{G}(s)\, \widetilde{\epsilon}(s)\,. \tag{2.7b}$$

This means that *the use of Laplace transforms allow us to write the creep and relaxation representations in a unique form, proper for each of them.*

In fact, Eq. (2.7a) is deduced from (2.5a) or (2.6a) according to

$$\widetilde{\epsilon}(s) = \sigma(0^+)\, \widetilde{J}(s) + \widetilde{J}(s)\, [s\, \tilde{\sigma}(s) - \sigma(0^+)] = J_g\, \tilde{\sigma}(s) + [s\, \widetilde{J}(s) - J_g]\, \tilde{\sigma}(s)\,,$$

[1]See later in Chapters 4 and 5 when we will use the non-dimensional memory functions

$$\Psi(t) := \frac{1}{J_g}\, \frac{dJ}{dt}\,, \quad \Phi(t) := \frac{1}{G_g}\, \frac{dG}{dt}\,.$$

and, similarly, Eq. (2.7b) is deduced from (2.5b) or (2.6b) according to

$$\tilde{\sigma}(s) = \epsilon(0^+)\,\widetilde{G}(s) + \widetilde{G}(s)\,[s\,\widetilde{\epsilon}(s) - \epsilon(0^+)] = G_g\,\widetilde{\epsilon}(s) + [s\,\widetilde{G}(s) - G_g]\,\widetilde{\epsilon}(s)\,.$$

We notice that (2.7b) is valid also if $G_g = \infty$, provided that we use a more general approach to the Laplace transform, based on the theory of generalized functions, see e.g. [Doetsch (1974); Ghizzetti and Ossicini (1971); Zemanian (1972)].

2.3 The four types of viscoelasticity

Since the creep and relaxation integral formulations must agree with each other, there must be a one-to-one correspondence between the relaxation modulus and the creep compliance. The basic relation between $J(t)$ and $G(t)$ is found noticing the following *reciprocity relation* in the Laplace domain, deduced from Eqs. (2.7a) (2.7b),

$$s\,\widetilde{J}(s) = \frac{1}{s\,\widetilde{G}(s)} \iff \widetilde{J}(s)\,\widetilde{G}(s) = \frac{1}{s^2}\,. \tag{2.8}$$

Indeed, inverting the R.H.S. of (2.8), we obtain

$$J(t)\,*\,G(t) := \int_0^t J(t-\tau)\,G(\tau)\,d\tau = t\,. \tag{2.9}$$

We can also obtain (2.8) noticing that, if the strain causal history is $J(t)$, then the stress response is $\Theta(t)$, the unit step function, so Eqs. (2.4a) and (2.5a) give

$$\Theta(t) = \int_{0^-}^t G(t-\tau)\,dJ(\tau) = J_g\,G(t) + \int_0^t G(t-\tau)\,\dot{J}(\tau)\,d\tau\,. \tag{2.10}$$

Then, applying the Laplace transform to (2.10) yields

$$\frac{1}{s} = J_g\,\widetilde{G}(s) + \widetilde{G}(s)\,\left[s\widetilde{J}(s) - J_g\right]\,.$$

Following [Pipkin (1986)], Eq. (2.10) allows us to obtain some notable relations in the time domain (inequalities and integral equations) concerning the material functions. Taking it for granted that, for $0 \le t \le +\infty$, $J(t)$ is non-negative and increasing, and $G(t)$ is non-negative and decreasing, Eq. (2.10) yields

$$1 = \int_{0^-}^{t} G(t-\tau)\, dJ(\tau) \ge G(t) \int_{0^-}^{t} dJ(\tau) = G(t)\, J(t)\,,$$

namely

$$J(t)\, G(t) \le 1\,. \tag{2.11}$$

We also note that if $J_g \neq 0$, we can rearrange (2.10) as a Volterra integral equation of the second kind, treating $G(t)$ as the unknown and $J(t)$ as the known function,

$$G(t) = J_g^{-1} - J_g^{-1} \int_0^t \dot{J}(t-\tau)\, G(\tau)\, d\tau\,. \tag{2.12a}$$

Similarly, if $G(t)$ is given and $G_g \neq \infty$, the equation for $J(t)$ is

$$J(t) = G_g^{-1} - G_g^{-1} \int_0^t \dot{G}(t-\tau)\, J(\tau)\, d\tau\,. \tag{2.12b}$$

Pipkin has also pointed out the following inequalities

$$G(t) \int_0^t J(\tau)\, d\tau \le t \le J(t) \int_0^t G(\tau)\, d\tau\,. \tag{2.13}$$

One of these inequalities (L.H.S.) is not as close as (2.11); the other (R.H.S.) gives new information. Furthermore, using with the due care the limiting theorems for the Laplace transform

$$f(0^+) = \lim_{s\to\infty} s\widetilde{f}(s)\,, \quad f(+\infty) = \lim_{s\to 0} s\widetilde{f}(s)\,,$$

we can deduce from the L.H.S of (2.8) that

$$J_g = \frac{1}{G_g}\,, \quad J_e = \frac{1}{G_e}\,, \tag{2.14}$$

with the convention that 0 and $+\infty$ are reciprocal to each other.

The remarkable relations allow us to classify the viscoelastic bodies according to their instantaneous and equilibrium responses. In fact, from Eqs. (2.2), (2.3) and (2.14) we easily recognize four possibilities for the limiting values of the creep compliance and relaxation modulus, as listed in Table 2.1.

Type	J_g	J_e	G_g	G_e
I	> 0	$< \infty$	$< \infty$	> 0
II	> 0	$= \infty$	$< \infty$	$= 0$
III	$= 0$	$< \infty$	$= \infty$	> 0
IV	$= 0$	$= \infty$	$= \infty$	$= 0$

Table 2.1 The four types of viscoelasticity.

We note that the viscoelastic bodies of type I exhibit both instantaneous and equilibrium elasticity, so their behaviour appears close to the purely elastic one for sufficiently short and long times. The bodies of type II and IV exhibit a complete stress relaxation (at constant strain) since $G_e = 0$ and an infinite strain creep (at constant stress) since $J_e = \infty$, so they do not show equilibrium elasticity. Finally, the bodies of type III and IV do not show instantaneous elasticity since $J_g = 0$ $(G_g = \infty)$.

Other properties will be pointed out later on.

2.4 The classical mechanical models

To get some feeling for linear viscoelastic behaviour, it is useful to consider the simpler behaviour of analog *mechanical models*. They are constructed from linear springs and dashpots, disposed singly and in branches of two (in series or in parallel) as it is shown in Fig. 2.1.

As analog of stress and strain, we use the total extending force and the total extension, respectively. We note that when two elements are combined in series [in parallel], their compliances [moduli] are additive. This can be stated as a combination rule: *creep compliances add in series, while relaxation moduli add in parallel.*

The important role in the literature of the mechanical models is justified by the historical development. In fact, the early theories were established with the aid of these models, which are still helpful to visualize properties and laws of the general theory, using the combination rule.

Now, it is worthwhile to consider the simple models of Fig. 2.1

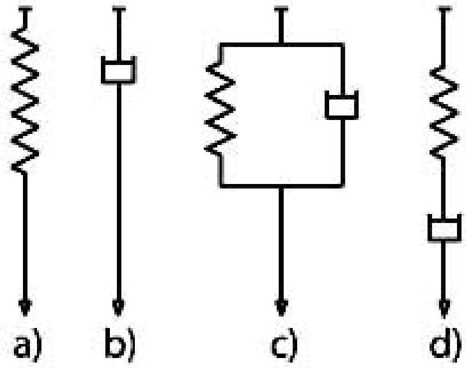

Fig. 2.1 The representations of the basic mechanical models: a) spring for Hooke, b) dashpot for Newton, c) spring and dashpot in parallel for Voigt, d) spring and dashpot in series for Maxwell.

by providing their governing stress–strain relations along with the related material functions.

The Hooke model. The spring a) in Fig. 2.1 is the elastic (or storage) element, as for it the force is proportional to the extension; it represents a perfect elastic body obeying the Hooke law. This model is thus referred to as the *Hooke model*. If we denote by m the pertinent elastic modulus we have

$$Hooke\ model\ :\ \sigma(t) = m\,\epsilon(t)\,, \tag{2.15a}$$

so

$$\begin{cases} J(t) = 1/m\,, \\ G(t) = m\,. \end{cases} \tag{2.15b}$$

In this case we have no creep nor relaxation so that the creep compliance and the relaxation modulus are constant functions: $J(t) \equiv J_g \equiv J_e = 1/m$; $G(t) \equiv G_g \equiv G_e = m$.

The Newton model. The dashpot b) in Fig. 2.1 is the viscous (or dissipative) element, the force being proportional to rate of extension; it represents a perfectly viscous body obeying the Newton law. This model is thus referred to as the *Newton model*. Denoting by b_1 the pertinent viscosity coefficient, we have

$$Newton\ model\ :\ \sigma(t) = b_1\,\frac{d\epsilon}{dt} \tag{2.16a}$$

so

$$\begin{cases} J(t) = \dfrac{t}{b_1}\,, \\ G(t) = b_1\,\delta(t)\,. \end{cases} \tag{2.16b}$$

In this case we have a linear creep $J(t) = J_+ t$ and instantaneous relaxation $G(t) = G_-\,\delta(t)$ with $G_- = 1/J_+ = b_1$.

We note that the Hooke and Newton models represent the limiting cases of viscoelastic bodies of type I and IV, respectively.

The Voigt model. A branch constituted by a spring in parallel with a dashpot is known as the *Voigt model*, c) in Fig. 2.1. We have

$$Voigt\ model\ :\ \sigma(t) = m\,\epsilon(t) + b_1\,\frac{d\epsilon}{dt}\,, \tag{2.17a}$$

so

$$\begin{cases} J(t) = J_1\left(1 - \mathrm{e}^{-t/\tau_\epsilon}\right), & J_1 = \dfrac{1}{m}\,,\ \tau_\epsilon = \dfrac{b_1}{m}\,, \\ G(t) = G_e + G_-\,\delta(t)\,, & G_e = m\,,\ G_- = b_1\,, \end{cases} \tag{2.17b}$$

where τ_ϵ is referred to as the *retardation time.*

The Maxwell model. A branch constituted by a spring in series with a dashpot is known as the *Maxwell model*, d) in Fig.2.1. We have

$$Maxwell\ model\ :\ \sigma(t) + a_1\,\frac{d\sigma}{dt} = b_1\,\frac{d\epsilon}{dt}\,, \tag{2.18a}$$

so

$$\begin{cases} J(t) = J_g + J_+\,t\,, & J_g = \dfrac{a_1}{b_1}\,,\ J_+ = \dfrac{1}{b_1}\,, \\ G(t) = G_1\,\mathrm{e}^{-t/\tau_\sigma}\,, & G_1 = \dfrac{b_1}{a_1}\,,\ \tau_\sigma = a_1\,, \end{cases} \tag{2.18b}$$

where τ_σ is is referred to as the *the relaxation time.*

The Voigt and the Maxwell models are thus the simplest viscoelastic bodies of type III and II, respectively. The Voigt model exhibits an exponential (reversible) strain creep but no stress relaxation; it is also referred as the retardation element. The Maxwell model exhibits an exponential (reversible) stress relaxation and a linear (non reversible) strain creep; it is also referred to as the relaxation element.

Based on the combination rule introduced above, we can continue the previous procedure in order to construct the simplest models of type I and IV that require three parameters.

The Zener model. The simplest viscoelastic body of type I is obtained by adding a spring either in series to a Voigt model or in parallel to a Maxwell model, respectively. In this way, according to the combination rule, we add a positive constant both to the Voigt-like creep compliance and to the Maxwell-like relaxation modulus so that $J_g > 0$ and $G_e > 0$. Such a model was considered by Zener [Zener (1948)] with the denomination of *Standard Linear Solid* (*S.L.S.*) and will be referred here also as the *Zener model*. We have

$$\textit{Zener model} \;:\; \left[1 + a_1 \frac{d}{dt}\right] \sigma(t) = \left[m + b_1 \frac{d}{dt}\right] \epsilon(t) \,, \qquad (2.19a)$$

so

$$\begin{cases} J(t) = J_g + J_1 \left(1 - \mathrm{e}^{-t/\tau_\epsilon}\right), & J_g = \dfrac{a_1}{b_1},\; J_1 = \dfrac{1}{m} - \dfrac{a_1}{b_1},\; \tau_\epsilon = \dfrac{b_1}{m}, \\ G(t) = G_e + G_1 \, \mathrm{e}^{-t/\tau_\sigma}, & G_e = m,\; G_1 = \dfrac{b_1}{a_1} - m,\; \tau_\sigma = a_1 \,. \end{cases} \qquad (2.19b)$$

We point out the condition $0 < m < b_1/a_1$ in order J_1, G_1 be positive and hence $0 < J_g < J_e < \infty$ and $0 < G_e < G_g < \infty$. As a consequence, we note that, for the *S.L.S.* model, the retardation time must be greater than the relaxation time, i.e. $0 < \tau_\sigma < \tau_\epsilon < \infty$.

The anti-Zener model. The simplest viscoelastic body of type IV requires three parameters, i.e. a_1, b_1, b_2; it is obtained by adding a dashpot either in series to a Voigt model or in parallel to a Maxwell model (Fig. 2.1c and Fig 2.1d, respectively). According to the combination rule, we add a linear term to the Voigt-like creep compliance and a delta impulsive term to the Maxwell-like relaxation modulus so that $J_e = \infty$ and $G_g = \infty$. We may refer to this model as the *anti-Zener model*. We have

$$\textit{anti-Zener model} \;:\; \left[1 + a_1 \frac{d}{dt}\right] \sigma(t) = \left[b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2}\right] \epsilon(t), \quad (2.20a)$$

so

$$\begin{cases} J(t) = J_+ t + J_1 \left(1 - \mathrm{e}^{-t/\tau_\epsilon}\right), & J_+ = \dfrac{1}{b_1},\; J_1 = \dfrac{a_1}{b_1} - \dfrac{b_2}{b_1^2},\; \tau_\epsilon = \dfrac{b_2}{b_1}, \\ G(t) = G_- \, \delta(t) + G_1 \, \mathrm{e}^{-t/\tau_\sigma}, & G_- = \dfrac{b_2}{a_1},\; G_1 = \dfrac{b_1}{a_1} - \dfrac{b_2}{a_1^2},\; \tau_\sigma = a_1. \end{cases} \qquad (2.20b)$$

We point out the condition $0 < b_2/b_1 < a_1$ in order J_1, G_1 to be positive. As a consequence we note that, for the *anti-Zener* model, the relaxation time must be greater than the retardation time, i.e. $0 < \tau_\epsilon < \tau_\sigma < \infty$, on the contrary of the Zener ($S.L.S.$) model.

In Fig. 2.2 we exhibit the mechanical representations of the Zener model [a), b)] and the anti-Zener model [c), d)]. Because of their main characteristics, these models can be referred as the *three-element elastic model* and the *three-element viscous model*, respectively.

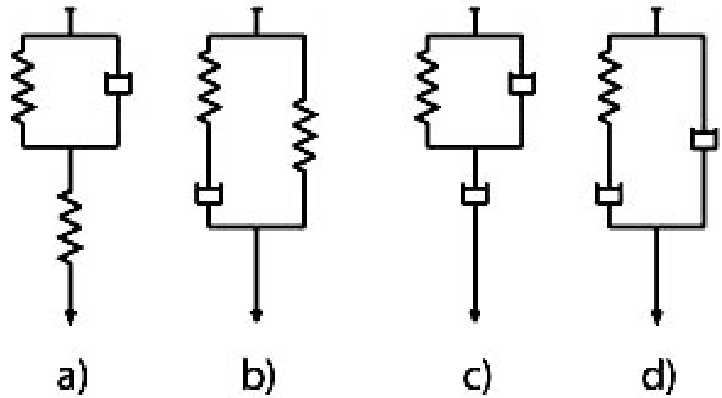

Fig. 2.2 The mechanical representations of the Zener [a), b)] and anti-Zener[c), d)] models: a) spring in series with Voigt, b) spring in parallel with Maxwell, c) dashpot in series with Voigt, d) dashpot in parallel with Maxwell.

By using the combination rule, general mechanical models can obtained whose material functions turn out to be of the type

$$\begin{cases} J(t) = J_g + \sum_n J_n \left(1 - \mathrm{e}^{-t/\tau_{\epsilon,n}}\right) + J_+ \, t \,, \\ G(t) = G_e + \sum_n G_n \, \mathrm{e}^{-t/\tau_{\sigma,n}} + G_- \, \delta(t) \,, \end{cases} \tag{2.21}$$

where all the coefficient are non negative. We note that the four types of viscoelasticity of Table 2.1 are obtained from Eqs. (2.21) by taking into account that

$$\begin{cases} J_e < \infty \iff J_+ = 0 \,, \quad J_e = \infty \iff J_+ \neq 0 \,, \\ G_g < \infty \iff G_- = 0 \,, \quad G_g = \infty \iff G_- \neq 0 \,. \end{cases} \tag{2.22}$$

The canonic forms. In Fig. 2.3, following [Gross (1953)], we exhibit the general mechanical representations of Eqs. (2.21) in terms of springs and dashpots (illustrated here by boxes), so summarizing the four *canonic forms.*

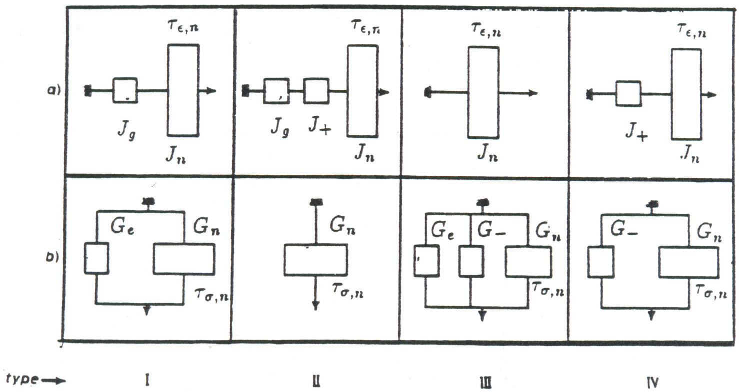

Fig. 2.3 The four types of canonic forms for the mechanical models: a) in creep representation; b) in relaxation representation.

The reader must note that in Fig. 2.3 the boxes denoted by J_g, G_e represent springs, those denoted by J_+, G_- represent dashpots and those denoted by $\{J_n, \tau_{\epsilon,n}\}$ and by $\{G_n, \tau_{\sigma,n}\}$ represent a sequence of Voigt models connected in series (*compound Voigt model*) and a sequence of Maxwell models connected in parallel (*compound Maxwell model*), respectively. The compound Voigt and Maxwell models are represented in Fig. 2.4.

As a matter of fact, each of the two representations can assume one of the *four canonic forms*, which are obtained by cutting out one, both, or none of the two single elements which have appeared besides the branches. Each of these four forms corresponds to each of the four types of linear viscoelastic behaviour (indicated in Table 2.1).

We recall that these material functions $J(t)$ and $G(t)$ are interrelated because of the *reciprocity relation* (2.8) in the Laplace domain.

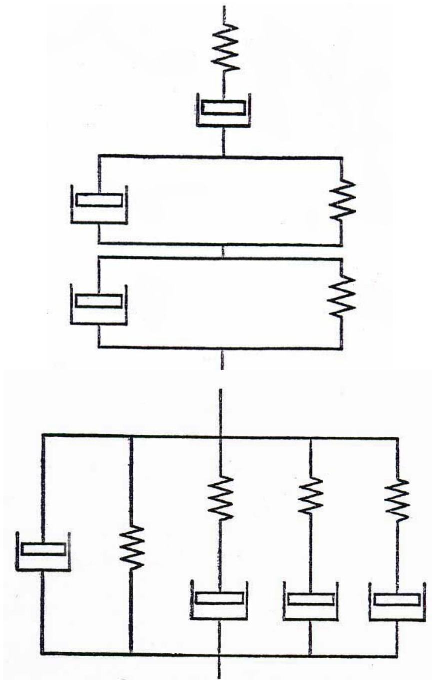

Fig. 2.4 The mechanical representations of the compound Voigt model (top) and compound Maxwell model (bottom).

Appealing to the theory of Laplace transforms, we get from (2.21)

$$\begin{cases} s\,\widetilde{J}(s) = J_g + \displaystyle\sum_n \frac{J_n}{1+s\,\tau_{\epsilon,n}} + \frac{J_+}{s}\,, \\ s\,\widetilde{G}(s) = G_e + \displaystyle\sum_n \frac{G_n\,(s\,\tau_{\sigma,n})}{1+s\,\tau_{\sigma,n}} + G_-\,s\,. \end{cases} \tag{2.23}$$

The second equality can be re-written as

$$s\,\widetilde{G}(s) = (G_e+\beta) - \sum_n \frac{G_n}{1+s\,\tau_{\sigma,n}} + G_-\,s\,,\ \text{with } \beta := \sum_n G_n\,.$$

Therefore, as a consequence of (2.23), $s\,\widetilde{J}(s)$ and $s\,\widetilde{G}(s)$ turn out to be *rational* functions in $\mathbb{C}$ with simple poles and zeros on the negative real axis $Re[s] < 0$ and, possibly, with a simple pole or with a simple zero at $s = 0$, respectively. As a consequence, see e.g.

[Bland (1960)], the above functions can be written as

$$s\,\widetilde{J}(s) = \frac{1}{s\,\widetilde{G}(s)} = \frac{P(s)}{Q(s)}\,, \text{ where } \begin{cases} P(s) = 1 + \displaystyle\sum_{k=1}^{p} a_k\, s^k\,, \\ Q(s) = m + \displaystyle\sum_{k=1}^{q} b_k\, s^k\,, \end{cases} \tag{2.24}$$

where the orders of the polynomials are equal ($q = p$) or differ of unity ($q = p + 1$) and the zeros are alternating on the negative real axis. The least zero in absolute magnitude is a zero of $Q(s)$. The ratio of any coefficient in $P(s)$ to any coefficient in $Q(s)$ is positive. The four types of viscoelasticity then correspond to whether the least zero is ($J_+ \neq 0$) or is not ($J_+ = 0$) equal to zero and to whether the greatest zero in absolute magnitude is a zero of $P(s)$ ($J_g \neq 0$) or a zero of $Q(s)$ ($J_g = 0$). We also point out that the polynomials at the numerator and denominator are *Hurwitz polynomials*, in that they have no zeros for $Re[s] > 0$, with $m \geq 0$ and $q = p$ or $q = p + 1$. Furthermore, the resulting rational functions $s\,\widetilde{J}(s)$, $s\,\widetilde{G}(s)$ turn out to be *positive real functions* in $\mathbb{C}$, namely they assume positive real values for $s \in \mathbb{R}^+$.

The operator equation. According to the classical theory of viscoelasticity (see e.g. [Alfrey (1948); Gross (1953)]), the above properties mean that the stress–strain relation must be a linear differential equation with constant (positive) coefficients of the following form

$$\left[1 + \sum_{k=1}^{p} a_k \frac{d^k}{dt^k}\right] \sigma(t) = \left[m + \sum_{k=1}^{q} b_k \frac{d^k}{dt^k}\right] \epsilon(t)\,. \tag{2.25}$$

Eq. (2.25) is referred to as the *operator equation* of the mechanical models, of which we have investigated the most simple cases illustrated in Figs. 2.1, 2.2. Of course, the constants m, a_k, b_k are expected to be subjected to proper restrictions in order to meet the physical requirements of realizability. For further details we refer the interested reader to [Hanyga (2005a); (2005b); (2005c)].

In Table 2.2 we summarize the four cases, which are expected to occur in the *operator equation* (2.25), corresponding to the four types of viscoelasticity.

Type	Order	m	J_g	G_e	J_+	G_-
I	$q = p$	> 0	a_p/b_p	m	0	0
II	$q = p$	$= 0$	a_p/b_p	0	$1/b_1$	0
III	$q = p + 1$	> 0	0	m	0	b_q/a_p
IV	$q = p + 1$	$= 0$	0	0	$1/b_1$	b_q/a_p

Table 2.2 The four cases of the operator equation.

We recognize that for $p = 1$ Eq. (2.25) includes the operator equations for the classical models with two parameters: Voigt and Maxwell; and with three parameters: Zener and anti-Zener. In fact, we recover the Voigt model (type III) for $m > 0$ and $p = 0, q = 1$, the Maxwell model (type II) for $m = 0$ and $p = q = 1$, the Zener model (type I) for $m > 0$ and $p = q = 1$, and the anti-Zener model (type IV) for $m = 0$ and $p = 1, q = 2$.

The Burgers model. With four parameters we can construct two models, the former with $m = 0$ and $p = q = 2$, the latter with $m > 0$ and $p = 1, q = 2$, referred in [Bland (1960)] to as four-element models of the first kind and of the second kind, respectively.

We restrict our attention to the former model, known as *Burgers model*, because it has found numerous applications, specially in geosciences, see e.g. [Klausner (1991); Carcione *et al.* (2006)]. We note that such a model is obtained by adding a dashpot or a spring to the representations of the Zener or of the anti-Zener model, respectively. Assuming the creep representation the dashpot or the spring is added in series, so the Burgers model results in a series combination of a Maxwell element with a Voigt element. Assuming the relaxation representation, the dashpot or the spring is added in parallel, so the Burgers model results in two Maxwell elements disposed in parallel. We refer the reader to Fig. 2.5 for the two mechanical representations of the Burgers model.

According to our general classification, the Burgers model is thus a four-element model of type II, defined by the four parameters $\{a_1, a_2, b_1, b_2\}$.

We have

$$Burgers\ model: \left[1 + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2}\right] \sigma(t) = \left[b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2}\right] \epsilon(t), \quad (2.26a)$$

so

$$\begin{cases} J(t) = J_g + J_+\, t + J_1 \left(1 - \mathrm{e}^{-t/\tau_\epsilon}\right), \\ G(t) = G_1\, \mathrm{e}^{-t/\tau_{\sigma,1}} + G_2\, \mathrm{e}^{-t/\tau_{\sigma,2}}. \end{cases} \quad (2.26b)$$

We leave to the reader to express as an exercise the physical quantities J_g, J_+, τ_ϵ and G_1, $\tau_{\sigma,1}$, G_2, $\tau_{\sigma,2}$, in terms of the four parameters $\{a_1, a_2, b_1, b_2\}$ in the operator equation (2.26a).

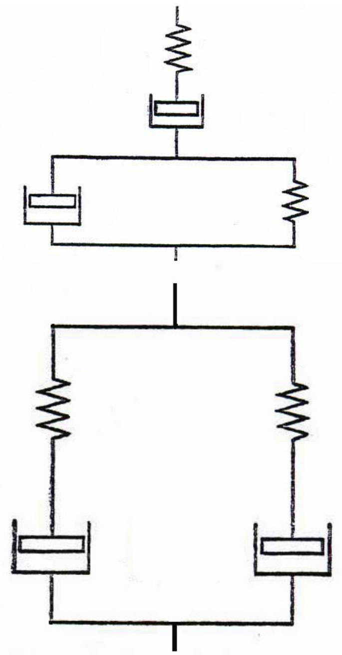

Fig. 2.5 The mechanical representations of the Burgers model: the creep representation (top), the relaxation representation (bottom).

Remark on the initial conditions :

We note that the initial conditions at $t = 0^+$ for the stress $\sigma(t)$ and strain $\epsilon(t)$,

$$\{\sigma^{(h)}(0^+)\,,\ h = 0, 1, \ldots p-1\}\,,\ \{\epsilon^{(k)}(0^+)\,,\ k = 0, 1, \ldots q-1\}\,,$$

do not appear in the operator equation, but they are required to be compatible with the integral equations (2.5a) and (2.5b) and consequently with the corresponding Laplace transforms provided by Eqs. (2.7a) and (2.7b). Since the above equations do not contain the initial conditions, some compatibility conditions at $t = 0^+$ must be *implicitly* required both for stress and strain. In other words, the equivalence between the integral equations (2.5a) and (2.5b), and the differential operator equation (2.25), implies that when we apply the Laplace transform to both sides of Eq. (2.25) the contributions from the initial conditions do not appear, namely they are vanishing or cancel in pair-balance. This can be easily checked for the simplest classical models described by Eqs. (2.17)–(2.20). For simple examples, let us consider the Voigt model for which $p = 0$, $q = 1$ and $m > 0$, see Eq. (2.17a), and the Maxwell model for which $p = q = 1$ and $m = 0$, see Eq. (2.18a).

For the Voigt model we get

$$s\widetilde{\sigma}(s) = m\widetilde{\epsilon}(s) + b_1\left[s\widetilde{\epsilon}(s) - \epsilon(0^+)\right]\,,$$

so, for any causal stress and strain histories, it would be

$$s\widetilde{J}(s) = \frac{1}{m + b_1 s} \iff \epsilon(0^+) = 0\,. \tag{2.27a}$$

We note that the condition $\epsilon(0^+) = 0$ is surely satisfied for any reasonable stress history since $J_g = 0$, but is not valid for any reasonable strain history; in fact, if we consider the relaxation test for which $\epsilon(t) = \Theta(t)$ we have $\epsilon(0^+) = 1$. This fact may be understood recalling that for the Voigt model we have $J_g = 0$ and $G_g = \infty$ (due to the delta contribution in the relaxation modulus).

For the Maxwell model we get

$$\widetilde{\sigma}(s) + a_1\left[s\widetilde{\sigma}(s) - \sigma(0^+)\right] = b_1\left[s\widetilde{\epsilon}(s) - \epsilon(0^+)\right]\,,$$

so, for any causal stress and strain histories it would be

$$s\widetilde{J}(s) = \frac{a_1}{b_1} + \frac{1}{b_1 s} \iff a_1\sigma(0^+) = b_1\epsilon(0^+)\,. \tag{2.27b}$$

We now note that the condition $a_1\sigma(0^+) = b_1\epsilon(0^+)$ is surely satisfied for any causal history, both in stress and in strain. This fact may be understood recalling that, for the Maxwell model, we have $J_g > 0$ and $G_g = 1/J_g > 0$.

Then we can generalize the above considerations stating that the compatibility relations of the initial conditions are valid for all the four types of viscoelasticity, as far as the creep representation is considered. When the relaxation representation is considered, caution is required for the types III and IV, for which, for correctness, we would use the generalized theory of integral transforms suitable just for dealing with generalized functions.

2.5 The time - and frequency - spectral functions

From the previous analysis of the classical mechanical models in terms of a finite number of basic elements, one is led to consider two *discrete* distributions of characteristic times (the *retardation* and the *relaxation* times), as it has been stated in Eq. (2.21). However, in more general cases, it is natural to presume the presence of *continuous* distributions, so that, for a viscoelastic body, the material functions turn out to be of the following form

$$\begin{cases} J(t) = J_g + a \int_0^\infty R_\epsilon(\tau)\, \left(1 - \mathrm{e}^{-t/\tau}\right)\, d\tau + J_+\, t\,, \\ G(t) = G_e + b \int_0^\infty R_\sigma(\tau)\, \mathrm{e}^{-t/\tau}\, d\tau + G_-\, \delta(t)\,, \end{cases} \tag{2.28}$$

where all the coefficients and functions are non-negative. The function $R_\epsilon(\tau)$ is referred to as the *retardation spectrum* while $R_\sigma(\tau)$ as the *relaxation spectrum*. For the sake of convenience we shall omit the suffix to denote any one of the two spectra; we shall refer to $R(\tau)$ as the *time–spectral function* in $\mathbb{R}^+$, with the supplementary normalization condition $\int_0^\infty R(\tau)\, d\tau = 1$ if the integral of $R(\tau)$ in $\mathbb{R}^+$ is convergent.

The discrete distributions of the classical mechanical models, see Eqs. (2.21), can be easily recovered from Eqs. (2.28). In fact, assuming $a \neq 0$, $b \neq 0$, we get after a proper use of the *delta-Dirac* *generalized functions*

$$\begin{cases} R_\epsilon(\tau) = \frac{1}{a} \sum_n J_n \, \delta(\tau - \tau_{\epsilon,n}) \,, & a = \sum_n J_n \,, \\ R_\sigma(\tau) = \frac{1}{b} \sum_n G_n \, \delta(\tau - \tau_{\sigma,n}) \,, & b = \sum_n G_n \,. \end{cases} \tag{2.29}$$

We now devote particular attention to the time-dependent contributions to the material functions (2.28) which are provided by the continuous or discrete spectra using for them the notation

$$\begin{cases} J_\tau(t) := a \int_0^\infty R_\epsilon(\tau) \left(1 - \mathrm{e}^{-t/\tau}\right) \, d\tau \,, \\ G_\tau(t) := b \int_0^\infty R_\sigma(\tau) \, \mathrm{e}^{-t/\tau} \, d\tau \,. \end{cases} \tag{2.30}$$

We recognize that $J_\tau(t)$ (that we refer as the *creep function with spectrum*) is a non-decreasing, non-negative function in $\mathbb{R}^+$ with limiting values $J_\tau(0^+) = 0$, $J_\tau(+\infty) = a$ or ∞, whereas $G_\tau(t)$ (that we refer as the *relaxation function with spectrum*) is a non-increasing, non-negative function in $\mathbb{R}^+$ with limiting values $G_\tau(0^+) = b$ or ∞, $G_\tau(+\infty) = 0$. More precisely, in view of the spectral representations (2.30), we have

$$\begin{cases} J_\tau(t) \ge 0 \,, \quad (-1)^n \, \frac{d^n J_\tau}{dt^n} \le 0 \,, \\ G_\tau(t) \ge 0 \,, \quad (-1)^n \, \frac{d^n G_\tau}{dt^n} \ge 0 \,. \end{cases} \quad t \ge 0 \,, \quad n = 1, 2, \ldots \,. \tag{2.31}$$

Using a proper terminology of mathematical analysis, see e.g. [Berg and Forst (1975); Feller (1971); Gripenberg *et al.* (1990)], $G_\tau(t)$ is a *completely monotonic function* whereas $J_\tau(t)$ is a *Bernstein function*, since it is a non-negative function with a completely monotonic derivative. These properties have been investigated by several authors, including [Molinari (1975)], [Del Piero and Deseri (1995)] and recently, in a detailed way, by [Hanyga (2005a); Hanyga (2005b); Hanyga (2005c)].

The determination of the *time–spectral functions* starting from the knowledge of the creep and relaxation functions is a problem that can be formally solved through the Titchmarsh inversion formula of the Laplace transform theory according to [Gross (1953)]. For this purpose let us recall the Gross method of *Laplace integral pairs*, which is based on the introduction of the *frequency–spectral functions* $S_\epsilon(\gamma)$ and $S_\sigma(\gamma)$ defined as

$$S_\epsilon(\gamma) := a\,\frac{R_\epsilon(1/\gamma)}{\gamma^2}\,, \quad S_\sigma(\gamma) := b\,\frac{R_\sigma(1/\gamma)}{\gamma^2}\,, \tag{2.32}$$

where $\gamma = 1/\tau$ denotes a retardation or relaxation frequency. We note that with the above choice it turns out

$$a\,R_\epsilon(\tau)\,d\tau = S_\epsilon(\gamma)\,d\gamma\,, \quad b\,R_\sigma(\tau)\,d\tau = S_\sigma(\gamma)\,d\gamma\,. \tag{2.33}$$

Differentiating (2.30) with respect to time yields

$$\begin{cases} \dot{J}_\tau(t) = a\displaystyle\int_0^\infty \frac{R_\epsilon(\tau)}{\tau}\,\mathrm{e}^{-t/\tau}\,d\tau = \int_0^\infty \gamma\,S_\epsilon(\gamma)\,\mathrm{e}^{-t\gamma}\,d\gamma\,, \\ \\ -\dot{G}_\tau(t) = b\displaystyle\int_0^\infty \frac{R_\sigma(\tau)}{\tau}\,\mathrm{e}^{-t/\tau}\,d\tau = \int_0^\infty \gamma\,S_\sigma(\gamma)\,\mathrm{e}^{-t\gamma}\,d\gamma\,. \end{cases} \tag{2.34}$$

We recognize that $\gamma\,S_\epsilon(\gamma)$ and $\gamma\,S_\sigma(\gamma)$ turn out to be the inverse Laplace transforms of $\dot{J}_\tau(t)$ and $-\dot{G}_\tau(t)$, respectively, where t is now considered the Laplace transform variable instead of the usual s. Adopting the usual notation for the Laplace transform pairs, we thus write

$$\begin{cases} \gamma\,S_\epsilon(\gamma) = a\,\dfrac{R_\epsilon(1/\gamma)}{\gamma} \div \dot{J}_\tau(t)\,, \\ -\gamma\,S_\sigma(\gamma) = b\,\dfrac{R_\sigma(1/\gamma)}{\gamma} \div \dot{G}_\tau(t)\,. \end{cases} \tag{2.35}$$

Consequently, when the *creep* and *relaxation* functions are given as analytical expressions, the corresponding frequency distributions can be derived by standard methods for the inversion of Laplace transforms; then, by using Eq. (2.32), the time–spectral functions can be easily derived.

Incidentally, we note that in the expressions defining the time and frequency spectra, often $d(\log\tau)$ and $d(\log\gamma)$ rather than $d\tau$ and $d\gamma$ are involved in the integrals. This choice changes the scaling of the above spectra in order to better deal with phenomena occurring on several time (or frequency) scales. In fact, introducing the new variables $u = \log\tau$ and $v = \log\gamma$, where $-\infty < u\,, v < +\infty$, the new spectra are related to the old ones as it follows

$$\hat{R}(u)\,du = R(\tau)\,\tau\,d\tau\,, \qquad \hat{S}(v)\,dv = S(\gamma)\,\gamma\,d\gamma\,. \tag{2.36}$$

Example of time and frequency spectra. As an example of spectrum determination, we now consider the creep function

$$J_\tau(t) = a\,\mathrm{Ein}\,(t/\tau_0)\,, \quad a > 0\,, \quad \tau_0 > 0\,, \tag{2.37}$$

where Ein denotes the modified exponential integral function, defined in the complex plane as an entire function whose integral and series representations read

$$\mathrm{Ein}\,(z) = \int_0^z \frac{1 - \mathrm{e}^{-\zeta}}{\zeta}\, d\zeta = -\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n\, n!}\,, \quad z \in \mathbb{C}\,. \tag{2.38}$$

For more details, see Appendix D. As a consequence we get

$$\frac{dJ_\tau}{dt}(t) = a\, \frac{1 - \mathrm{e}^{-\gamma_0 t}}{t}\,, \quad \gamma_0 = \frac{1}{\tau_0}\,. \tag{2.39}$$

By inspection of a table of Laplace transform pairs we get the inversion and, using (2.35), the following time and frequency–spectra

$$R_\epsilon(\tau) = \begin{cases} 0\,, & 0 < \tau < \tau_0\,, \\ 1/\tau\,, & \tau_0 < \tau < \infty\,; \end{cases} \tag{2.40a}$$

$$S_\epsilon(\gamma) = \begin{cases} a/\gamma\,, & 0 < \gamma < \gamma_0\,, \\ 0\,, & \gamma_0 < \gamma < \infty\,. \end{cases} \tag{2.40b}$$

Plotted against $\log \tau$ and $\log \gamma$ the above spectra are step-wise distributions.

Stieltjes transforms. To conclude this section, following [Gross (1953)], we look for the relationship between the Laplace transform of the creep/relaxation function and the corresponding time or frequency spectral function. If we choose the frequency spectral function, we expect that a sort of iterated Laplace transform be involved in the required relationship, in view of the above results. In fact, applying the Laplace transform to Eq. (2.34) we obtain

$$\begin{cases} s\widetilde{J_\tau}(s) = \displaystyle\int_0^\infty \frac{\gamma\, S_\epsilon(\gamma)}{s + \gamma}\, d\gamma\,, \\[2ex] s\widetilde{G_\tau}(s) = -\displaystyle\int_0^\infty \frac{\gamma\, S_\sigma(\gamma)}{s + \gamma}\, d\gamma + G_\tau(0^+)\,. \end{cases} \tag{2.41}$$

Introducing the function

$$\widetilde{L}(s) = \int_0^\infty \frac{\gamma\, S(\gamma)}{s+\gamma}\, d\gamma\,, \tag{2.42}$$

where the suffix ϵ or σ is understood, we recognize that $\widetilde{L}(s)$ is the Stieltjes transform of $\gamma\, S(\gamma)$. The inversion of the Stieltjes transform may be carried out by Titchmarsh's formula,

$$\gamma\, S(\gamma) = \frac{1}{\pi}\, Im\left\{\widetilde{L}\left(\gamma\, \mathrm{e}^{-i\pi}\right)\right\} = \frac{1}{\pi}\lim_{\delta\to 0} Im\left\{\widetilde{L}(-\gamma - i\delta)\right\}\,. \tag{2.43}$$

Consequently, when the Laplace transforms of the creep and relaxation functions are given as analytical expressions, the corresponding frequency distributions can be derived by standard methods for the inversion of Stieltjes transforms; then, by using Eq. (2.32) the time–spectral functions can be easily derived.

2.6 History in $\mathbb{R}$: the Fourier transform approach and the dynamic functions

In addition to the unit step (that is acting for $t \geq 0$), another widely used form of excitation in viscoelasticity is the *harmonic* or *sinusoidal* excitation that is acting for all of $\mathbb{R}$ since it is considered (ideally) applied since $t = -\infty$. The corresponding responses, which are usually referred to as the *dynamic functions*, provide, together with the material functions previously investigated, a complete description of the viscoelastic behaviour. In fact, according to [Findley *et al.* (1976)], creep and relaxation experiments provide information starting from a lower limit of time which is approximatively of $10\, s$, while dynamic experiments with sinusoidal excitations may provide data from about $10^{-8}\, s$ to about $10^3\, s$. Thus there is an overlapping region ($10\, s - 10^3\, s$) where data can be obtained from both types of experiments. Furthermore, the dynamic experiments provide information about storage and dissipation of the mechanical energy, as we shall see later.

In the following the basic concepts related to sinusoidal excitations is introduced. It is convenient to use the complex notation for sinusoidal functions, i.e. the excitations in stress and strain in non-dimensional form are written as

$$\sigma(t;\omega) = \mathrm{e}^{\,i\omega t}\,,\ \epsilon(t;\omega) = \mathrm{e}^{\,i\omega t}\,,\ \omega > 0\,,\ -\infty < t < +\infty\,, \tag{2.44}$$

where ω denotes the angular frequency ($f = \omega/2\pi$ is the cyclic frequency and $T = 1/f$ is the period). Of course, in Eq. (2.44) we understand to take the real or imaginary part of the exponential in view of the Euler formula $\mathrm{e}^{\pm i\omega t} = \cos\,\omega t \pm i\,\sin\,\omega t$.

For histories of type (2.44), the integral stress–strain relations (2.4) can be used to provide the corresponding response functions. We obtain, after an obvious change of variable in the integrals,

$$\sigma(t) = \mathrm{e}^{\,i\omega t} \Longrightarrow \epsilon(t) = J^*(\omega)\,\mathrm{e}^{\,i\omega t}\,,\ J^*(\omega) := i\omega\,\widehat{J}(\omega)\,, \tag{2.45a}$$

and

$$\epsilon(t) = \mathrm{e}^{\,i\omega t} \Longrightarrow \sigma(t) = G^*(\omega)\,\mathrm{e}^{i\omega t}\,,\ G^*(\omega) := i\omega\,\widehat{G}(\omega)\,, \tag{2.45b}$$

where $\widehat{J}(\omega) = \int_0^\infty J(t)\,\mathrm{e}^{\,-i\omega t}\,dt$ and $\widehat{G}(\omega) = \int_0^\infty G(t)\,\mathrm{e}^{\,-i\omega t}\,dt.$

The functions $J^*(\omega)$ and $G^*(\omega)$ are usually referred as the *complex compliance* and *complex modulus*, respectively, or, simply, the *dynamic functions* of the viscoelastic body. They are related with the Fourier transforms of the causal functions $J(t)$ and $G(t)$ and therefore can be expressed in terms of their Laplace transforms as follows

$$J^*(\omega) = s\,\widetilde{J}(s)\Big|_{s=i\omega}\,, \quad G^*(\omega) = s\,\widetilde{G}(s)\Big|_{s=i\omega}\,, \tag{2.46}$$

so that, in agreement with the reciprocity relation (2.8),

$$J^*(\omega)\,G^*(\omega) = 1\,. \tag{2.47}$$

2.7 Storage and dissipation of energy: the loss tangent

Introducing the *phase shift* $\delta(\omega)$ between the sinusoidal excitation and the sinusoidal response in Eqs. (2.45a) and (2.45b), we can write

$$J^*(\omega) = J'(\omega) - iJ''(\omega) = |J^*(\omega)|\,\mathrm{e}^{\,-i\delta(\omega)}\,, \tag{2.48a}$$

and

$$G^*(\omega) = G'(\omega) + iG''(\omega) = |G^*(\omega)|\,\mathrm{e}^{\,+i\delta(\omega)}\,. \tag{2.48b}$$

As a consequence of energy considerations, recalled hereafter by following [Tschoegel (1989)], it turns out that $\delta(\omega)$ must be positive (in particular, $0 < \delta(\omega) < \pi/2$) as well as the quantities $J'(\omega)$, $J''(\omega)$ and $G'(\omega)$, $G''(\omega)$ entering Eqs. (2.48a) and (2.48b). Usually, J' and G' are called the *storage compliance* and the*storage modulus*, respectively, while J'' and G'' are called the *loss compliance* and the *loss modulus*, respectively; as we shall see, the above attributes connote something to do with energy storage and loss. Furthermore,

$$\tan\,\delta(\omega) = \frac{J''(\omega)}{J'(\omega)} = \frac{G''(\omega)}{G'(\omega)} \tag{2.49}$$

is referred to as the *loss tangent*, a quantity that summarizes the damping ability of a viscoelastic body, as we will show explicitly below.

During the deformation of a viscoelastic body, part of the total work of deformation is dissipated as heat through viscous losses, but the remainder of the deformation-energy is stored elastically. It is frequently of interest to determine, for a given sample of material in a given mode of deformation, the total work of deformation as well as the amount of energy stored and the amount dissipated. Similarly, one may wish to know the rate at which the energy of deformation is absorbed by the material or the rate at which it is stored or dissipated.

The rate at which energy is absorbed per unit volume of a viscoelastic material during deformation is equal to the *stress power*, i.e. the rate at which work is performed. The stress power at time t is

$$\dot{W}(t) = \sigma(t)\,\dot{\epsilon}(t)\,, \tag{2.50}$$

i.e. it is the product of the instantaneous stress and rate of strain. The electrical analog of (2.50) is the well-known relation which states that the electrical power equals the product of instantaneous voltage and current. The total work of deformation or, in other words, the mechanical energy absorbed per unit volume of material in the deformation from the initial time t_0 up to the current time t, results in

$$W(t) = \int_{t_0}^{t} \dot{W}(\tau)\,d\tau = \int_{t_0}^{t} \sigma(t)\,\dot{\epsilon}(\tau)\,d\tau\,. \tag{2.51}$$

Assuming the possibility of computing separately the energy stored, $W_s(t)$, and the energy dissipated, $W_d(t)$, we can write

$$W(t) = W_s(t) + W_d(t)\,, \qquad \dot{W}(t) = \dot{W}_s(t) + \dot{W}_d(t)\,. \tag{2.52}$$

Please note that all energy or work terms and their derivatives will henceforth refer to unit volume of the material even when this is not explicitly stated.

Elastically stored energy is potential energy. Energy can also be stored inertially as kinetic energy. Such energy storage may be encountered in fast loading experiments, e.g. in response to impulsive excitation, or in wave propagation at high frequency. In the linear theory of viscoelastic behaviour, however, inertial energy storage plays no role.

How much of the total energy is stored and how much is dissipated, i.e. the precise form of (2.52), depends, of course, on the nature of the material on the one hand, and on the type of deformation on the other. The combination of stored and dissipated energy is conveniently based on the representation of linear viscoelastic behaviour by models (the classical mechanical models) in that, by definition, the energy is dissipated uniquely in the dashpots and stored uniquely in the springs.

For the rate of energy dissipation we get

$$\dot{W}_d(t) = \sum_n \sigma_{dn}(t)\,\dot{\epsilon}_{dn}(t) = \sum_n \eta_n(t)\,[\dot{\epsilon}_{dn}(t)]^2\,, \tag{2.53}$$

where $\sigma_{dn}(t)$ and $\dot{\epsilon}_{dn}(t)$ are the stress and the rate of strain, respectively, in the n-th dashpot, which are related by the equality $\sigma_{dn}(t) = \eta_n\,\dot{\epsilon}_{dn}(t)$ with η_n denoting the coefficient of viscosity.

For the energy storage we get

$$\begin{aligned} W_s(t) &= \sum_n \int_{t_0}^{t} \sigma_{sn}(\tau)\,\dot{\epsilon}_{sn}(\tau)\,d\tau \\ &= \sum_n G_n \int_{\epsilon_{sn}(0)}^{\epsilon_{sn}(t)} \epsilon_{sn}(\tau)\,d\epsilon_{sn}(\tau) \\ &= \frac{1}{2} \sum_n G_n\,[\epsilon_{sn}(t)]^2\,, \end{aligned} \tag{2.54}$$

where $\sigma_{sn}(t)$ and $\epsilon_{sn}(t)$ are the stress and the strain, respectively, in the n-th spring, which are related by the equality $\sigma_{sn}(t) = G_n \, \epsilon_{sn}(t)$ with G_n denoting the elastic modulus.

Equations (2.53) and (2.54) are the basic relations for determining energy storage and dissipation, respectively, during a particular deformation. They are given meaning by finding the stresses, strains, rates of strain in the springs and dashpots of mechanical models in the given mode of deformation. The nature of the material is reflected in the distribution of the parameters G_n and η_n. Examples have been given by [Tschoegel (1989)], to which the interested reader is referred. In the absence of appropriate spring-dashpot models we may still think of energy-storing and energy-dissipating mechanisms but without identifying them with mechanical models, and modify the arguments as needed.

Let us now compute the total energy $W(t)$ and its rate $\dot{W}(t)$ for sinusoidal excitations, and possibly determine the corresponding contributions due to the storing and dissipating mechanisms [Tschoegel (1989)]. Taking the imaginary parts in (2.45b) we have

$$\epsilon(t) = \sin \omega t \implies \sigma(t) = G'(\omega) \, \sin \omega t + G''(\omega) \, \cos \omega t \,, \qquad (2.55)$$

where the terms on the R.H.S. represent, respectively, the components of the stress which are in phase and out of phase with the strain.

Since the rate of strain is $\omega \, \cos \omega t$, Eqs. (2.50) and (2.55) lead to

$$\dot{W}(t) = \frac{\omega}{2} \left[G'(\omega) \, \sin 2\omega t + G''(\omega) \, (1 + \cos 2\omega t) \right] . \qquad (2.56)$$

Integration of (2.56), subject to the initial condition $W(0) = 0$, yields

$$W(t) = \frac{1}{4} \left[G'(\omega) \, (1 - \cos 2\omega t) + G''(\omega) \, (2\omega t + \sin 2\omega t) \right] . \qquad (2.57)$$

In general, all storing mechanisms are not in phase as well as all dissipating mechanisms, so that in Eqs. (2.56) and (2.57) we cannot recognize the partial contributions to the storage and dissipation of energy. Only if *phase coherence* is assumed among the energy storing mechanisms on the one hand and the energy dissipating mechanisms on the other, we can easily separate the energy stored from that dissipated. We get

$$\dot{W}_s^c(t) = \frac{\omega}{2} \, G'(\omega) \, \sin 2\omega t \,, \; \dot{W}_d^c(t) = \frac{\omega}{2} \, G''(\omega) \, (1 + \cos 2\omega t) \,, \quad (2.58)$$

hence

$$W_s^c(t) = \frac{G'(\omega)}{4}\,(1-\cos 2\omega t)\,,\;\; W_d^c(t) = \frac{G''(\omega)}{4}\,(2\omega t+\sin 2\omega t)\,, \tag{2.59}$$

where the superscript c points out the hypothesis of coherence.

For the stored energy, a useful parameter is the *average* taken over a full cycle of the excitation. We find from (2.59)

$$\begin{aligned} \langle W_s(\omega)\rangle &:= \frac{1}{T}\int_t^{t+T} W_s^c(\tau)\,d\tau \\ &= \frac{\omega\, G'(\omega)}{8\pi}\int_0^{2\pi/\omega}(1-\cos 2\omega\tau)\,d\tau = \frac{G'(\omega)}{4}\,, \end{aligned} \tag{2.60}$$

which is one half of the maximum coherently storable energy.

For the dissipated energy we consider the amount of energy that would be dissipated coherently over a full cycle of the excitation. We find from (2.56)

$$\begin{aligned} \Delta W_d(\omega) &:= \int_t^{t+T} \dot{W}_d^c(\tau)\,d\tau \\ &= \frac{\omega\, G''(\omega)}{2}\int_0^{2\pi/\omega}(1+\cos 2\omega\tau)\,d\tau = \pi\, G''(\omega)\,. \end{aligned} \tag{2.61}$$

We recognize that Eqs. (2.60) and (2.61) justify the names of $G'(\omega)$ and $G''(\omega)$ as *storage* and *loss modulus*, respectively.

Usually the dissipation in a viscoelastic medium is measured by introducing the so-called *specific dissipation function*, or *internal friction*, defined as

$$Q^{-1}(\omega) = \frac{1}{2\pi}\,\frac{\Delta W_d}{W_s^*}\,, \tag{2.62}$$

where ΔW_d is the amount of energy dissipated coherently in one cycle and W_s^* is the peak energy stored coherently during the cycle. It is worthwhile to note that Q^{-1} denotes the reciprocal of the so-called *quality factor*, that is denoted by Q in electrical engineering, see e.g. [Knopoff (1956)]. From Eqs. (2.49) and (2.60) and (2.62) it turns out that

$$Q^{-1}(\omega) = \tan\delta(\omega)\,. \tag{2.63}$$

This equation shows that the damping ability of a linear viscoelastic body is dependent only on the tangent of the phase angle, namely the *loss tangent* introduced in Eq. (2.49), that is a function of frequency and is a measure of a physical property, but is independent of the stress and strain amplitudes.

2.8 The dynamic functions for the mechanical models

Let us conclude this chapter with the evaluation of the dynamic functions (complex moduli or complex compliances) for the classical mechanical models as it can be derived from their expressions according to Eq. (2.49), with special emphasis to their loss tangent.

For convenience, let us consider the Zener model, that contains as limiting cases the Voigt and Maxwell models, whereas we leave as an exercise the evaluation of the dynamic functions for the anti-Zener and Burgers models.

For this purpose we consider the dynamic functions, namely the complex compliance $J^*(\omega)$ and the complex modulus $G^*(\omega)$, for the Zener model, that can be derived from the Laplace transforms of the corresponding material functions $J(t)$ and $G(t)$ according to Eqs. (2.46). Using Eqs. (2.19a) and (2.19b) we get

$$J^*(\omega) = s\,\widetilde{J}(s)\Big|_{s=i\omega} = J_g + J_1 \frac{1}{1+s\tau_\epsilon}\Big|_{s=i\omega}, \tag{2.64}$$

$$G^*(\omega) = s\,\widetilde{G}(s)\Big|_{s=i\omega} = G_e + G_1 \frac{s\tau_\sigma}{1+s\tau_\sigma}\Big|_{s=i\omega}. \tag{2.65}$$

Then we get:

$$J^*(\omega) = J'(\omega) - J''(\omega)\,, \quad \begin{cases} J'(\omega) = J_g + J_1 \dfrac{1}{1+\omega^2\tau_\epsilon^2}\,, \\ J''(\omega) = J_1 \dfrac{\omega\tau_\epsilon}{1+\omega^2\tau_\epsilon^2}\,; \end{cases} \tag{2.66}$$

$$G^*(\omega) = G'(\omega) + G''(\omega)\,, \quad \begin{cases} G'(\omega) = G_e + G_1 \dfrac{\omega\tau_\sigma}{1+\omega^2\tau_\sigma^2}\,, \\ G''(\omega) = G_1 \dfrac{\omega^2\tau_\sigma^2}{1+\omega^2\tau_\sigma^2}\,. \end{cases} \tag{2.67}$$

Taking into account the definitions in (2.19b) that provide the interrelations among the constants in Eqs. (2.64) and (2.67), we find it convenient to introduce a new characteristic time

$$\tau := \sqrt{\tau_\sigma \, \tau_\epsilon} \,, \tag{2.68}$$

and

$$\Delta := \frac{\tau_\epsilon - \tau_\sigma}{\tau} = \begin{cases} \dfrac{J_e - J_g}{\sqrt{J_g \, J_e}} \,, \\[2ex] \dfrac{G_g - G_e}{\sqrt{G_g \, G_e}} \,. \end{cases} \tag{2.69}$$

Then, after simple algebraic manipulations, the loss tangent for the Zener model turns out to be

$$\textit{Zener model} \ : \ \tan\,\delta(\omega) = \frac{J''(\omega)}{J'(\omega)} = \frac{G''(\omega)}{G'(\omega)} = \Delta \, \frac{\omega\,\tau}{1 + (\omega\,\tau)^2} \,. \tag{2.70}$$

We easily recognize that the loss tangent for the Zener model attains its maximum value $\Delta/2$ for $\omega = 1/\tau$.

It is instructive to adopt another notation in order to provide alternative expressions (consistent with the results by [Caputo and Mainardi (1971b)]), by introducing the characteristic frequencies related to the retardation and relaxation times:

$$\begin{cases} \alpha := 1/\tau_\epsilon = m/b_1 \,, \\ \beta := 1/\tau_\sigma = 1/a_1 \,, \end{cases} \quad \text{with} \quad 0 < \alpha < \beta < \infty \,. \tag{2.71}$$

As a consequence the constitutive equations (2.19a) and (2.19b) for the Zener model read

$$\left[1 + \frac{1}{\beta}\,\frac{d}{dt}\right] \sigma(t) = m \left[1 + \frac{1}{\alpha}\,\frac{d}{dt}\right] \epsilon(t) \,, \quad m = G_e = G_g \, \frac{\alpha}{\beta} \,. \tag{2.72}$$

Then, limiting ourselves to consider the complex modulus, this reads

$$G^*(\omega) = G_e \frac{1 + i\omega/\alpha}{1 + i\omega/\beta} = G_g \frac{\alpha + i\omega}{\beta + i\omega} \,, \tag{2.73}$$

henceforth

$$G^*(\omega) = G'(\omega) + G''(\omega) \,, \quad \begin{cases} G'(\omega) = G_g \dfrac{\omega^2 + \alpha\,\beta}{\omega^2 + \beta^2} \,, \\[2ex] G''(\omega) = G_g \dfrac{\omega(\beta - \alpha)}{\omega^2 + \beta^2} \,. \end{cases} \tag{2.74}$$

Finally, the loss tangent turns out to be

$$\textit{Zener model} \;:\quad \tan\delta(\omega) = \frac{G''(\omega)}{G'(\omega)} = (\beta-\alpha)\,\frac{\omega}{\omega^2+\alpha\,\beta}\,. \tag{2.75}$$

Now the loss tangent attains its maximum value $(\beta-\alpha)/(2\sqrt{\alpha\,\beta})$ for $\omega = \sqrt{\alpha\,\beta}$, a result consistent with that obtained with the previous notation.

It is instructive to plot in Fig. 2.6 the dynamic functions $G'(\omega)$, $G''(\omega)$ and the loss tangent $\tan\delta(\omega)$ versus ω for the Zener model. For convenience we use non-dimensional units and we adopt for ω a logarithmic scale from 10^{-2} to 10^2. We take $\alpha = 1/2$, $\beta = 2$ so $\alpha\beta \equiv 1$, and $G_g = 1$ so $G_e = \alpha\beta = 1/4$.

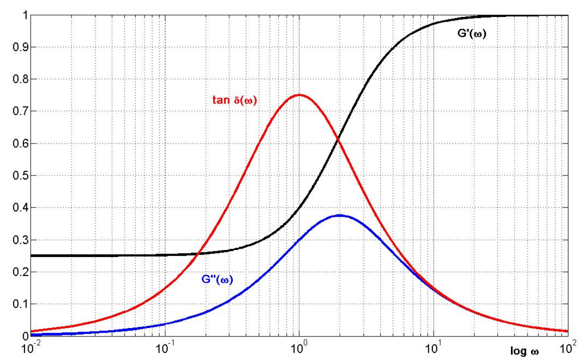

Fig. 2.6 Plots of the dynamic functions $G'(\omega)$, $G''(\omega)$ and loss tangent $\tan\delta(\omega)$ versus $\log\omega$ for the Zener model.

As expected, from Eq. (2.75) we easily recover the expressions of the loss tangent for the limiting cases of the Hooke, Newton, Voigt and Maxwell models. We obtain:

$$\textit{Hooke model}\ (\alpha=\beta=0)\ :\ \tan\delta(\omega) = 0\,, \tag{2.76}$$

$$\textit{Newton model}\ (0=\alpha<\beta=\infty)\ :\ \tan\delta(\omega) = \infty\,, \tag{2.77}$$

$$\textit{Voigt model}\ (0<\alpha<\beta=\infty)\ :\ \tan\delta(\omega) = \frac{\omega}{\alpha} = \omega\,\tau_\epsilon\,, \tag{2.78}$$

$$\textit{Maxwell model}\ (0=\alpha<\beta<\infty)\ :\ \tan\delta(\omega) = \frac{\beta}{\omega} = \frac{1}{\omega\,\tau_\sigma}\,. \tag{2.79}$$

We recover that the Hooke model exhibits only energy storage whereas the Newton model, only energy dissipation. The Voigt and Maxwell models exhibit both storage and dissipation of energy, in such a way that their loss tangent turns out to be directly proportional and inversely proportional to the frequency, respectively. As a consequence, with respect to the loss tangent, the Zener model exhibits characteristics common to the Voigt and Maxell models in the extremal frequency regions: precisely, its loss tangent is increasing for very low frequencies (like for the Voigt model), is decreasing for very high frequencies (like for the Maxwell model), and attains its (finite) maximum value within an intermediate frequency range.

2.9 Notes

The approach to linear viscoelasticity based on memory functions (the "hereditary" approach) was started by V. Volterra, e.g. [Volterra (1913); Volterra (1928); Volterra (1959)] and pursued in Italy by a number of mathematicians, including: Cisotti, Giorgi, Graffi, Tricomi, Benvenuti, Fichera, Caputo, Fabrizio and Morro.

Many results of the Italian school along with the recent theoretical achievements of the "hereditary" approach are well considered in the book [Fabrizio and Morro (1992)] and in the papers [Deseri *et al.* (2006)], [Fabrizio *et al.* (2009)].

Our presentation is mostly based on our past review papers [Caputo and Mainardi (1971b); Mainardi (2002a)] and on classical books [Bland (1960); Gross (1953); Pipkin (1986); Tschoegel (1989)].

For the topic of *realizability of the viscoelastic models* and for the related concept of complete monotonicity the reader is referred to the papers by A. Hanyga, see e.g. [Hanyga (2005a); Hanyga (2005b); Hanyga (2005c)] and the references therein.

We have not considered (in the present edition) the topic of *ladder networks*: the interested reader is invited to consult the excellent treatise [Tschoegel (1989)] and the references therein. We note that in the literature of ladder networks, the pioneering contributions by the late Ellis Strick, Professor of Geophysics at the University of Pittsburgh, are unfortunately not mentioned: these contributions turn out to be hidden in his unpublished lecture notes [Strick (1976)].

To conclude, applications of the linear theory of viscoelasticity appear in several fields of material sciences such as chemistry (e.g. [Doi and Edwards (1986); Ferry (1980)], seismology (e.g. [Aki and Richards (1980); Carcione (2007)]), soil mechanics (e.g. [Klausner (1991)]), arterial rheology (e.g. [Craiem *et al.* 2008]), food rheology (e.g. [Rao and Steffe (1992)]), to mention just a few. Because papers are spread out in a large number of journals, any reference list cannot be exhaustive.